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On unsymmetrically impinging jets

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Two impinging two-dimensional incompressible inviscid fluid jets of known widths and velocities produce two outgoing jets. The speeds of the outgoing jets are readily determined from the Bernoulli equation. Their two widths and two directions (four quantities) are related by conservation of mass and conservation of two components of momentum (three relations). Because these three conservation relations do not suffice to determine the four unknowns, Milne-Thomson (1968) states on p. 302 that 'a unique solution is, in general, not possible'. He incorrectly attributes this indeterminateness to disregard of 'the initial conditions from which this steady motion is supposed to arise'.

The correct resolution of this problem is that one must specify the lateral positions of the incoming jets in addition to their speeds and directions. This is analogous to specifying the impact parameter in the collision of two particles. It yields a fourth relation which makes the number of equations equal to the number of unknowns.

To illustrate this point, we begin with Milne-Thomson's general result (p. 302, §11.35) for the complex position z of the free boundaries of the four jets in terms of a parameter θ, $0 \geqslant \theta \geqslant -2\pi$:

$$\pi z = \tfrac{1}{2}i(-h_2\alpha\,\mathrm{e}^{-\mathrm{i}\alpha}+k_1\beta\,\mathrm{e}^{-\mathrm{i}\beta}+k_2\gamma\,\mathrm{e}^{-\mathrm{i}\gamma})+h_1\log\sin\tfrac{1}{2}\theta+h_2\,\mathrm{e}^{-\mathrm{i}\alpha}\log\sin\tfrac{1}{2}(\theta+\alpha)$$
$$-k_1\,\mathrm{e}^{-\mathrm{i}\beta}\log\sin\tfrac{1}{2}(\theta+\beta)-k_2\,\mathrm{e}^{-\mathrm{i}\gamma}\log\sin\tfrac{1}{2}(\theta+\gamma). \quad (1)$$

Here h_1, h_2, k_1, k_2 are the widths and 0, $-\alpha$, $-\beta$, $-\gamma$ are the asymptotic directions of the velocities in the four jets. The coordinate system is chosen so that the flow is steady with the origin at the stagnation point. Then it follows that the flow speed is constant on the free boundaries. Thus all four jets have the same speed.

Conservation of mass and momentum yields three relations among the eight constants (p. 301):

$$h_1+h_2 = k_1+k_2, \quad (2)$$

$$h_1+h_2\cos\alpha = k_1\cos\beta+k_2\cos\gamma, \quad (3)$$

$$h_2\sin\alpha = k_1\sin\beta+k_2\sin\gamma. \quad (4)$$

In addition four quantities, the directions 0 and $-\alpha$ and the widths h_1 and h_2 of the incoming jets, must be specified. Then (2)–(4) are three equations for the four quantities k_1, k_2, β and γ.

A fourth relation can be obtained by specifying the vertical position of the asymptote to the upper surface of the jet coming in from the left. As θ tends to -2π from above, (1) shows that $x = \operatorname{Re} z$ tends to $-\infty$ and $y = \operatorname{Im} z$ tends to y_L given by

$$\pi y_\mathrm{L} = -h_2\sin\alpha\log\sin\tfrac{1}{2}\alpha+k_1\sin\beta\log\sin\tfrac{1}{2}\beta+k_2\sin\gamma\log\sin\tfrac{1}{2}\gamma$$
$$+\tfrac{1}{2}(-h_2\alpha\cos\alpha+k_1\beta\cos\beta+k_2\cos\beta). \quad (5)$$

When y_L is specified, (5) provides the desired fourth relation. Then (2)–(5) determine

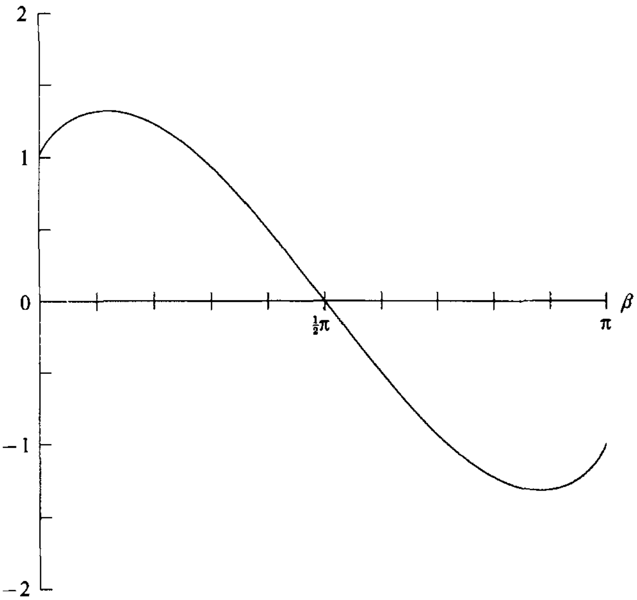

FIGURE 1. The offset $(y_R - y_L)/h$ versus the angle β of the outgoing jets, based upon equation (9). The ordinate is also $1 - 2y_L h^{-1}$, as (6) shows.

k_1, k_2, β and γ. Note that y_L is defined in a coordinate system with the stagnation point at the origin, so that y_L is the distance of the asymptote from the stagnation point.

To illustrate the use of these equations we consider the special case $h_1 = h_2 = h$ and $\alpha = \pi$ so the incoming jets are of equal widths and are oppositely directed. Then (2)–(4) show that $k_1 = k_2 = h$ and $\gamma = \beta + \pi$. Upon using these values in (5) we get

$$\frac{y_L}{h} = \tfrac{1}{2}(1 - \cos\beta) + \frac{1}{\pi}\sin\beta \log\tan\tfrac{1}{2}\beta, \quad 0 \leqslant \beta \leqslant \pi. \tag{6}$$

This is the asymptote to the upper surface of the jet incident from the left. The asymptote to its lower surface, obtained by letting θ increase to 0, is found to lie at the distance h below the upper one. This equation (6) determines y_L in terms of β, and it can be solved for β in terms of y_L. Figure 1 is a graph of $1 - 2y_L/h$ as a function of β based on (6).

The equation of the jet boundaries in this special case is, from (1),

$$\frac{\pi z}{h} = \tfrac{1}{2}i\pi(1 - e^{-i\beta}) + \log\tan\tfrac{1}{2}\theta - e^{-i\beta}\log\tan\tfrac{1}{2}(\theta + \beta). \tag{7}$$

Now we let θ tend to $-\pi$ from below in (7) to get the upper asymptote to the jet incident from the right

$$\frac{y_R}{h} = \tfrac{1}{2}(1 + \cos\beta) - \frac{1}{\pi}\sin\beta \log\tan\tfrac{1}{2}\beta, \quad 0 \leqslant \beta \leqslant \pi. \tag{8}$$

The asymptote to the lower surface of this jet is h below the upper one. Specifying any one of these four asymptotes determines β and thus the entire flow.

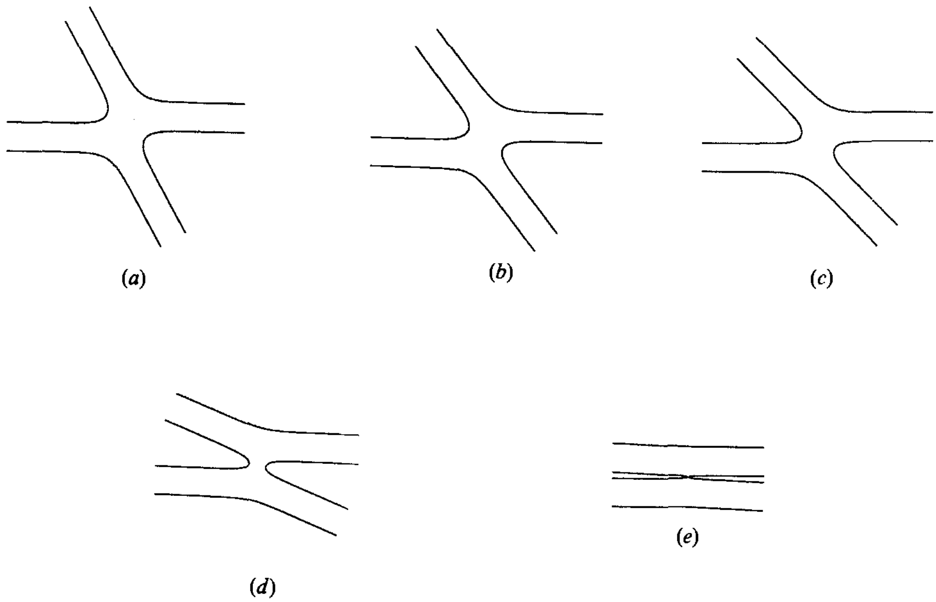

FIGURE 2. Impinging and outgoing jets for various values of β, based upon equation (7). The values of β and the offsets given by (9) are: (*a*) $\beta = \pi/3$, $(y_R - y_L)/h = 0.8518$; (*b*) $\beta = \pi/3.5452$, $(y_R - y_L)/h = 1$; (*c*) $\beta = \pi/4$, $(y_R - y_L)/h = 1.0386$, the same offset as in (*e*); (*d*) $\beta = \pi/8.518$, $(y_R - y_L)/h = 1.3181$, the maximum offset; (*e*) $\beta = \pi/73.336$, offset the same as for $\beta = \pi/4$, shown in (*c*).

It is more symmetrical to consider the impact of parameter or offset, $y_R - y_L$, between the upper asymptotes to the two incoming jets

$$\frac{1}{h}(y_R - y_L) = \cos\beta - \frac{2}{\pi}\sin\beta \log\tan\tfrac{1}{2}\beta, \quad 0 \leqslant \beta \leqslant \pi. \tag{9}$$

Figure 1 is also a graph of the right-hand side of (9). It shows that specifying $(y_R - y_L)/h$ yields a unique value of β when $|y_R - y_L|/h < 1$. The case of zero offset, for which $\beta = \frac{1}{2}\pi$, is the one presented by Milne-Thomson and previous authors.

Figure 1 also shows the surprising result that for a small range of offsets $1.318 > |y_R - y_L|/h \geqslant 1$ there are solutions, and in fact two for each offset in this range. These are in addition to the non-interacting jets that just pass by each other, which can occur when $|y_R - y_L| > h$. The corresponding range of β is $0 \leqslant \beta \leqslant \pi/3.545$ and $\pi - \pi/3.545 \leqslant \beta \leqslant \pi$. I conjecture that the interacting jets are unstable in this range.

Some examples of the jet boundaries for various values of β are shown in figure 2, based upon (7).

In the analysis of impinging three-dimensional jets, the shapes, lateral positions and orientations of the incident jets would have to be prescribed.

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